Problem 1

Show that $\mathbf{NP}$ is closed under union, concatenation, intersection, and Kleene star.

Problem 2

Show that the following language is $\mathbf{NP}$-complete (polynomial-time reduction from $\mathbf{SAT}$),

$$\text{DOUBLE SAT} = \{ \langle \varphi \rangle \mid \varphi \text{ is a Boolean formula with at least two satisfying assignments} \}.$$  

Problem 3

Let $G$ be a graph. A clique in $G$ is a subset $C$ of vertices such that any two vertices $u, v \in C$ are adjacent. A vertex cover of $G$ is a subset $S$ of vertices such that every edge of $G$ has at least one endpoint in $S$.

Show polynomial-time reductions between the following languages,

$$\text{CLIQUE} = \{ \langle G, k \rangle \mid G \text{ contains a clique of size } k \},$$
$$\text{VERTEX COVER} = \{ \langle G, \ell \rangle \mid G \text{ has a vertex cover of size } \ell \}.$$
Problem 4

This problem investigates resolution, a method for proving the unsatisfiability of CNF formulas. Let \( \varphi = C_1 \land C_2 \land \cdots \land C_m \) be a CNF formula with clauses \( C_1, \ldots, C_m \). Let

\[
C = \{ C_i \mid C_i \text{ is a clause of } \varphi \}.
\]

A resolution step proceeds as follows: Take two clauses \( C_a \) and \( C_b \) in \( C \), which both have some variable \( x \) occurring positively in one of the clauses and negatively in the other. Thus, \( C_a = (x \lor y_1 \lor y_2 \lor \cdots \lor y_k) \) and \( C_b = (\neg x \lor z_1 \lor z_2 \lor \cdots \lor z_\ell) \), where the \( y_i \) and the \( z_j \) are literals. Form the new clause \( (y_1 \lor y_2 \lor \cdots \lor y_k \lor z_1 \lor z_2 \lor \cdots \lor z_\ell) \) and remove repeated literals. Add this new clause to \( C \).

Repeat the resolution steps until no additional clauses can be obtained. If the empty clause \( () \) is in \( C \), then declare \( \varphi \) unsatisfiable.

Say that resolution is sound if it never declares satisfiable formulas to be unsatisfiable. Say that resolution is complete if all unsatisfiable formulas are declared to be unsatisfiable.

Part a

Show that resolution is sound and complete.

Part b

Use Part a to show that 2-CNF SAT is in \( \mathbf{P} \). (2-CNF SAT is the set of satisfiable formulas in conjunctive normal form with at most two literals per clause.)